ORBIT DETERMINATION

Giovanni Federico Gronchi
Dipartimento di Matematica, Università di Pisa , Largo B. Pontecorvo 5, 56127 Pisa, Italy

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Summary

To determine the orbit of a solar system body means to compute its position and velocity at a certain time using the observations of the body, e.g. right ascension and declination if we use an optical telescope. This allows us to compute ephemerides and predict the position of the body at different times.

This branch of Celestial Mechanics has attracted the interest of several scientists over the last centuries. However, the ongoing improvements of the observational technologies have set up new orbit determination problems in the recent years: this is partly due to the availability of different observables (e.g. the range, with radar telescopes), but also to the huge amount of data that can be collected. For these reasons scientists have been induced to think about new algorithms to compute orbits.

In this chapter we present a review of some orbit determination methods, with particular care about the computation of preliminary orbits. Here we include both classical methods, due to Gauss and Laplace, and very recent ones, which are suitable for the sets of optical observations made with modern telescopes. Also the problem of alternative solutions is considered: we describe some results on the geometric characterization of the number of preliminary solutions. The last part of this chapter is devoted to the linkage of short arcs, that is an identification problem appearing with the very large amount of observations that can be made with modern instruments.

1. Introduction

The determination of the orbits of the solar system bodies is an important branch of Celestial Mechanics and has attracted the interest of several scientists over the last centuries. The main problem can be formulated as follows: given a set of observable quantities of a celestial body, made at different epochs (e.g. the angular positions of an asteroid on the celestial sphere), compute the position and velocity of the body at the average time of the observations, so that it is possible to predict the position of the body in the future. The observations of a celestial body are affected by errors, e.g. due to the instruments, or to atmospheric effects. It is necessary to take into account the effect of these errors in an orbit determination procedure.

Here is a short (and incomplete) list of scientists who made important contributions to this field: E. Halley, A. J. Lexell, J. L. Lagrange, A. M. Legendre, F. F. Tisserand, P. S. De Laplace, C. F. Gauss, O. Mossotti, H. Poincaré, C. W. L. Charlier, A. Leuschner.

A key event for the development of orbit determination methods was the discovery of Ceres, the first *main belt* (The main belt asteroids (MBAs) are located between the orbits of Mars and Jupiter.) asteroid, by Giuseppe Piazzi (Observatory of Palermo, January 1, 1801). He could follow up Ceres in the sky for about 1 month, collecting about 20 observations. Then a problem was set up for the scientists of that epoch: to predict when and in which part of the sky Ceres could be observed again. Ceres was recovered one year later by H. W. Olbers and F. Von Zach, following the suggestions of C. F. Gauss, who among many other scientific interests, was attracted by astronomical problems and became the director of the Göttingen observatory in 1807.

Gauss' method consists in two steps: compute a preliminary orbit (see Section 2), then apply an iterative method to obtain a solution of a least squares fit (see Section 3). Unfortunately, there can be more than one preliminary orbit: this problem is addressed in Section 4.

At the beginning of the XIX century an asteroid was typically observed only once per night; moreover the number of objects that could be observed was much smaller. The observations at the present days are different: we can detect many more asteroids and we compare images of the same field taken a few minutes apart to search for moving objects. In Figure 1 we show three images of the detection of an asteroid in September 2002.

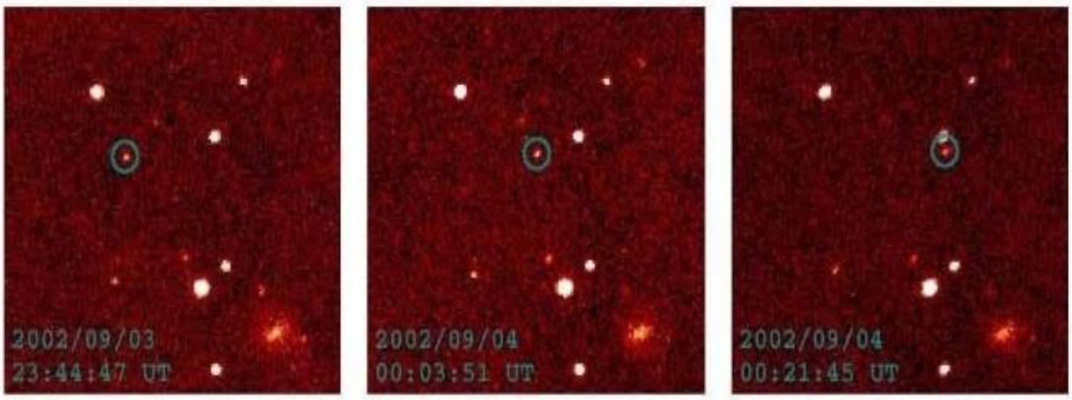

Figure 1. Three images showing the detection of an asteroid (encircled in the figures) during the night of September 3, 2002: the time interval between two consecutive images is 20 minutes. Courtesy of F. Bernardi.

Thus today there is also an identification problem, that is to join together sets of observations taken in different nights as belonging to the same observed object. The different cases occurring in the identification are described in Sections 5, 6.

There is a broad literature about orbit determination: here we restrict the exposition to the most famous classical methods and to some recent achievements concerning objects orbiting around the Sun (e.g. asteroids), observed with optical instruments.

2. Classical Methods of Preliminary Orbit Determination

We illustrate the two classical methods by Laplace and by Gauss to compute a preliminary orbit of a celestial body orbiting around the Sun and observed from the Earth.

2.1. Laplace's Method

Assume we have the observations (α_i, δ_i) of a solar system body at times t_i, $i = 1 \ldots m$, $m \geq 3$; then we can interpolate for $\alpha, \delta, \dot{\alpha}, \dot{\delta}$ at a mean time $\bar{t}$, where the dots indicate the time derivatives. To obtain an orbit we have to compute the radial distance ρ and the radial velocity $\dot{\rho}$ at the same time $\bar{t}$.

Let $\boldsymbol{\rho} = \rho \hat{\mathbf{e}}^{\rho}$ be the geocentric position vector of the observed body, with $\rho = \|\boldsymbol{\rho}\|$ and $\hat{\mathbf{e}}^{\rho} = (\cos\delta\cos\alpha, \cos\delta\sin\alpha, \sin\delta)$, where α, δ are the right ascension and declination. We denote by $\mathbf{q} = q\hat{\mathbf{q}}$ the heliocentric position of the center of the Earth, with $q = \|\mathbf{q}\|$ and by $\mathbf{r} = \mathbf{q} + \boldsymbol{\rho}$ the heliocentric position of the body.

We use the arc length s to parameterize the motion: s is related to the time t by

$$\frac{ds}{dt} = \sqrt{\dot{\alpha}\cos^2\delta + \dot{\delta}^2} \overset{\text{def}}{=} \eta \qquad (\text{proper motion}).$$

We introduce the moving orthonormal basis

$$\hat{\mathbf{e}}^{\rho}, \quad \hat{\mathbf{e}}^{v} = \frac{d\hat{\mathbf{e}}^{\rho}}{ds}, \quad \hat{\mathbf{e}}^{n} = \hat{\mathbf{e}}^{\rho} \times \hat{\mathbf{e}}^{v}. \tag{1}$$

The relation

$$\frac{d\,\hat{\mathbf{e}}^{v}}{ds} = -\hat{\mathbf{e}}^{\rho} + \kappa\hat{\mathbf{e}}^{n}$$

defines the geodesic curvature κ. The second derivative of ρ with respect to t can be written as

$$\frac{d^2\boldsymbol{\rho}}{dt^2} = \left(\ddot{\rho} - \rho\eta^2\right)\hat{\mathbf{e}}^{\rho} + \left(\rho\dot{\eta} + 2\dot{\rho}\eta\right)\hat{\mathbf{e}}^{v} + \left(\rho\eta^2\kappa\right)\hat{\mathbf{e}}^{n}.$$

On the other hand, assuming the asteroid and the Earth move on Keplerian orbits, we have

$$\frac{d^2\boldsymbol{\rho}}{dt^2} = \frac{d^2}{dt^2}\left(\mathbf{r} - \mathbf{q}\right) = -\frac{\mu}{r^3}\mathbf{r} + \frac{\mu + \mu_{\oplus}}{q^3}\mathbf{q},$$

with $r = \|\mathbf{r}\|$ and μ, $\mu_{\oplus}$ the masses of the Sun and of the Earth respectively.

Neglecting the mass of the Earth and projecting the equation of motion onto $\hat{\mathbf{e}}^n$ at time $\bar{t}$ we obtain the dynamical equation of Laplace's method

$$\mathcal{C}\frac{\rho}{q} = 1 - \frac{q^3}{r^3} \quad \text{with} \quad \mathcal{C} = \frac{\eta^2\kappa q^3}{\mu\left(\hat{\mathbf{q}} \cdot \hat{\mathbf{e}}^n\right)}, \tag{2}$$

where $\rho, q, r, \eta, \hat{\mathbf{q}}, \hat{\mathbf{e}}^n, \mathcal{C}$ denote the values of these quantities at time $\bar{t}$.

In Eq. (2) ρ and r are unknown, while the other quantities can be computed by interpolation. Using (2) and the geometric equation

$$r^2 = q^2 + \rho^2 + 2q\rho\cos\epsilon, \tag{3}$$

where $\cos\epsilon = \mathbf{q} \cdot \boldsymbol{\rho} / \left(q\rho\right)$, we can write a polynomial equation of degree eight for r at time $\bar{t}$ by eliminating the geocentric distance:

$$\mathcal{C}^2 r^8 - q^2\left(\mathcal{C}^2 + 2\mathcal{C}\cos\epsilon + 1\right)r^6 + 2q^5\left(\mathcal{C}\cos\epsilon + 1\right)r^3 - q^8 = 0. \tag{4}$$

The occurrence of alternative solutions in Eqs. (2), (3) is discussed in Section 1.

The projection of the equations of motion on $\hat{\mathbf{e}}^v$ gives

$$\rho\dot{\eta} + 2\dot{\rho}\eta = \mu\left(\mathbf{q} \cdot \hat{\mathbf{e}}^v\right)\left(\frac{1}{q^3} - \frac{1}{r^3}\right). \tag{5}$$

We can use Eq. (5) to compute $\dot{\rho}$ from the values of r, ρ found by (4) and (2).

2.2. Gauss' Method

Assume we have three observations $\left(\alpha_i, \delta_i\right)$, $i = 1, 2, 3$ of a solar system body at times

t_i, with $t_1 < t_2 < t_3$. Let $\mathbf{r}_i, \boldsymbol{\rho}_i$ denote the heliocentric and topocentric positions respectively of the body, and let $\mathbf{q}_i$ be the heliocentric position of the observer. Gauss' method uses the heliocentric positions

$$\mathbf{r}_i = \boldsymbol{\rho}_i + \mathbf{q}_i \quad i = 1,2,3. \tag{6}$$

We assume that $\left|t_i - t_j\right|, 1 \le i, j \le 3$, is much smaller than the period of the orbit and write $\mathcal{O}(\Delta t)$ for the order of magnitude of the time differences.

From the coplanarity condition we have

$$\lambda_1 \mathbf{r}_1 - \mathbf{r}_2 + \lambda_3 \mathbf{r}_3 = 0 \tag{7}$$

for $\lambda_1, \lambda_3 \in \mathbb{R}$. The vector product of both members of (7) with $\mathbf{r}_i, i = 1,3$ and the fact that the vectors $\mathbf{r}_i \times \mathbf{r}_j, i < j$ have all the same orientation as $c = \mathbf{r}_h \times \dot{\mathbf{r}}_h$, $h = 1,2,3$ implies

$$\lambda_1 = \frac{\mathbf{r}_2 \times \mathbf{r}_3 \cdot \mathbf{c}}{\mathbf{r}_1 \times \mathbf{r}_3 \cdot \mathbf{c}}, \quad \lambda_3 = \frac{\mathbf{r}_1 \times \mathbf{r}_2 \cdot \mathbf{c}}{\mathbf{r}_1 \times \mathbf{r}_3 \cdot \mathbf{c}}.$$

Let $\boldsymbol{\rho}_i = \rho_i \hat{\mathbf{e}}_i^{\rho}$, $i = 1,2,3$. From the scalar product of $\hat{\mathbf{e}}_1^{\rho} \times \hat{\mathbf{e}}_3^{\rho}$ with both members of (7), using (6), we obtain

$$\rho_2 \left[\hat{\mathbf{e}}_1^{\rho} \times \hat{\mathbf{e}}_2^{\rho} \cdot \hat{\mathbf{e}}_3^{\rho}\right] = \hat{\mathbf{e}}_1^{\rho} \times \hat{\mathbf{e}}_3^{\rho} \cdot \left[\lambda_1 \mathbf{q}_1 - \mathbf{q}_2 + \lambda_3 \mathbf{q}_3\right]. \tag{8}$$

The differences $\mathbf{r}_i - \mathbf{r}_2$, $i = 1,3$, are expanded in powers of $t_{ij} = t_i - t_j = \mathcal{O}(\Delta t)$ by the f, g series formalism; thus $\mathbf{r}_i = f_i \mathbf{r}_2 + g_i \dot{\mathbf{r}}_2$, with

$$f_i = 1 - \frac{\mu}{2} \frac{t_{i2}^2}{r_2^3} + \mathcal{O}\left(\Delta t^3\right), \qquad g_i = t_{i2}\left(1 - \frac{\mu}{6} \frac{t_{i2}^2}{r_2^3}\right) + \mathcal{O}\left(\Delta t^4\right). \tag{9}$$

Then $\mathbf{r}_i \times \mathbf{r}_2 = -g_i \mathbf{c}$, $\mathbf{r}_1 \times \mathbf{r}_3 = \left(f_1 g_3 - f_3 g_1\right)\mathbf{c}$ and

$$\lambda_1 = \frac{g_3}{f_1 g_3 - f_3 g_1}, \quad \lambda_3 = \frac{-g_1}{f_1 g_3 - f_3 g_1}, \tag{10}$$

$$f_1 g_3 - f_3 g_1 = t_{31}\left(1 - \frac{\mu}{6} \frac{t_{31}^2}{r_2^3}\right) + \mathcal{O}\left(\Delta t^4\right). \tag{11}$$

Using (9) and (11) in (10) we obtain

$$\lambda_1 = \frac{t_{32}}{t_{31}}\left[1 + \frac{\mu}{6r_2^3}\left(t_{31}^2 - t_{32}^2\right)\right] + \mathcal{O}\left(\Delta t^3\right), \tag{12}$$

$$\lambda_3 = \frac{t_{21}}{t_{31}}\left[1 + \frac{\mu}{6r_2^3}\left(t_{31}^2 - t_{21}^2\right)\right] + \mathcal{O}\left(\Delta t^3\right). \tag{13}$$

Let $V = \hat{\mathbf{e}}_1^{\rho} \times \hat{\mathbf{e}}_2^{\rho} \cdot \hat{\mathbf{e}}_3^{\rho}$. By substituting (12), (13) into (8), using relations $t_{31}^2 - t_{32}^2 = t_{21}\left(t_{31} + t_{32}\right)$ and $t_{31}^2 - t_{21}^2 = t_{32}\left(t_{31} + t_{21}\right)$, we can write

$$\begin{aligned} -V\rho_2 t_{31} &= \hat{\mathbf{e}}_1^{\rho} \times \hat{\mathbf{e}}_3^{\rho} \cdot \left(t_{32}\mathbf{q}_1 - t_{31}\mathbf{q}_2 + t_{21}\mathbf{q}_3\right) \\ &+ \hat{\mathbf{e}}_1^{\rho} \times \hat{\mathbf{e}}_3^{\rho} \cdot \left[\frac{\mu}{6r_2^3}\left[t_{32}t_{21}\left(t_{31} + t_{32}\right)\mathbf{q}_1 + t_{32}t_{21}\left(t_{31} + t_{21}\right)\mathbf{q}_3\right]\right] + \mathcal{O}\left(\Delta t^4\right) \end{aligned} \tag{14}$$

If the $\mathcal{O}\left(\Delta t^4\right)$ terms are neglected, the coefficient of $1/r_2^3$ in (14) is

$$B\left(\mathbf{q}_1, \mathbf{q}_3\right) = \frac{\mu}{6} t_{32}t_{21}\hat{\mathbf{e}}_1^{\rho} \times \hat{\mathbf{e}}_3^{\rho} \cdot \left[\left(t_{31} + t_{32}\right)\mathbf{q}_1 + \left(t_{31} + t_{21}\right)\mathbf{q}_3\right]. \tag{15}$$

Then multiply (14) by $q_2^3 / B\left(\mathbf{q}_1, \mathbf{q}_3\right)$ to obtain

$$-\frac{V\rho_2 t_{31}}{B\left(\mathbf{q}_1, \mathbf{q}_3\right)} q_2^3 = \frac{q_2^3}{r_2^3} + \frac{A\left(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\right)}{B\left(\mathbf{q}_1, \mathbf{q}_3\right)},$$

where

$$A\left(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\right) = q_2^3 \hat{\mathbf{e}}_1^{\rho} \times \hat{\mathbf{e}}_3^{\rho} \cdot \left[t_{32}\mathbf{q}_1 - t_{31}\mathbf{q}_2 + t_{21}\mathbf{q}_3\right].$$

Setting

$$\mathcal{C} = \frac{V t_{31} q_2^4}{B\left(\mathbf{q}_1, \mathbf{q}_3\right)}, \qquad \gamma = -\frac{A\left(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\right)}{B\left(\mathbf{q}_1, \mathbf{q}_3\right)}; \tag{16}$$

we obtain the dynamical equation of Gauss' method:

$$\mathcal{C} = \frac{\rho_2}{q_2} = \gamma - \frac{q_2^3}{r_2^3}. \tag{17}$$

After the possible values for r_2 have been found by (17) and by the geometric equation

$$r_2^2 = \rho_2^2 + q_2^2 + 2\rho_2 q_2 \cos\epsilon_2, \tag{18}$$

then the velocity vector $\dot{r}_2$ can be computed, e.g. from Gibbs' formulas.

The occurrence of alternative solutions of Eqs. (17), (18) is discussed in Section 2.

We observe that in his original formulation Gauss used different quantities as unknowns, whose values could be improved by an iterative procedure (today called Gauss map).

3. Least Squares Orbits

We consider the differential equation

$$\frac{d\mathbf{y}}{dt} = (\mathbf{y}, t, \boldsymbol{\mu}) \tag{19}$$

giving the state $\mathbf{y} \in \mathbb{R}^p$ of the system at time t (e.g. $p = 6$ if $\mathbf{y}$ is a vector of orbital elements). Here $\boldsymbol{\mu} \in \mathbb{R}^{p'}$ are some constants, called dynamical parameters.

The integral flow, solution of (19) for initial data $\mathbf{y}_0$ at time t_0, is denoted by $\Phi_{t_0}^t(\mathbf{y}_0, \boldsymbol{\mu})$.

We also introduce the *observation function*

$$\mathbf{R} = (R_1, \ldots, R_k), R_j = R_j(\mathbf{y}, t, \mathbf{v}), \quad j = 1 \ldots k$$

depending on the state $\mathbf{y}$ of the system at time t, and on some constants $\mathbf{v} \in \mathbb{R}^{p''}$, called kinematical parameters. Moreover we define the *prediction function* $\tilde{\mathbf{r}}(t)$ as the composition of the integral flow with the observation function:

$$\tilde{\mathbf{r}}(t) = \mathbf{R}\left(\boldsymbol{\Phi}_{t_0}^t(\mathbf{y}_0, \boldsymbol{\mu}), t, \mathbf{v}\right).$$

These functions gives a prediction for a specific observation at time t.

We can group the multidimensional data and predictions into two vectors. For example, assume the available observations at time τ_j are the right ascension α_j and the declination δ_j, for $j = 1, \ldots, h$. Then

$$m = kh, \quad k = 2, \quad t_{2j-1} = t_{2j} = \tau_j, \quad \begin{cases} r_{2j-1} = \alpha_j \\ r_{2j} = \delta_j \end{cases}, \qquad \begin{cases} r(t_{2j-1}) = \tilde{r}_1(\tau_j) \\ r(t_{2j}) = \tilde{r}_2(\tau_j) \end{cases}.$$ with

components r_i, $r(t_i)$ and define the vector of the residuals

$$\xi = (\xi_1 \dots \xi_m), \quad \xi_i = r_i - r(t_i), \qquad i = 1 \dots m.$$

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Bibliography

[1] A. Celletti, G. Pinzari (2005): 'Four Classical Methods for Determining Planetary Elliptic Elements: A Comparison', *CMDA* , 1–52 [Description of some classical orbit determination methods, including the ones of Gauss and Laplace. Here the Gauss map is described in full details.]

[2] A. Celletti, G. Pinzari (2006): 'Dependence on the observational time intervals and domain of convergence of orbital determination methods', *CMDA* , 327–344 [This paper investigates how the variation of the time intervals between the observations affects the determination of the orbits using the classical methods. Sufficient conditions for the convergence of Gauss' method are also introduced.]

[3] C. V. L. Charlier (1910): 'On Multiple Solutions in the Determination of Orbits from three Observation', *MNRAS* , 120–124 [Geometric interpretation of the occurrence of alternative solutions in preliminary orbit determination from three observations. Here the observations are assumed to be geocentric.]

[4] C. V. L. Charlier (1911): 'Second Note on Multiple Solutions in the Determination of Orbits from three Observation', *MNRAS* , 454–459 [Additional study of alternative solutions and introduction of the singular curve.]

[5] G. Foderà Serio, A. Manara, P. Sicoli (2003): '*Giuseppe Piazzi and the Discovery of Ceres*', in Asteroids III, W.F. Bottke Jr. et al. eds., Arizona University Press [The history of the discovery of the asteroid Ceres, and of the determination of its orbit by Gauss.]

[6] C. F. Gauss (1809): '*Theoria motus corporum coelestium in sectionibus conicis Solem ambientium*', '*Theory of the Motion of the Heavenly Bodies Moving about the Sun in Conic Sections*', reprinted by Dover publications (1963) [Source reference for Gauss' method, introducing preliminary orbits, and their refinement by iterative methods. The least squares method is also presented here.]

[7] G. F. Gronchi (2009): 'Multiple Solutions in Preliminary Orbit Determination from Three Observations', *CMDA* /4, 301–326 [Generalization of Charlier's theory taking into account topocentric observations.]

[8] G. F. Gronchi, L. Dimare, A. Milani (2010): 'Orbit determination with the two-body integrals', *CMDA* /3, 299–318 [Introduction of algebraic methods to solve the linkage equations with the angular momentum and energy integrals.]

[9] G. F. Gronchi, D. Farnocchia, L. Dimare (2011): 'Orbit determination with the two-body integrals. II', *CMDA* /3, 257–270 [Algebraic methods applied to different linkage equations, with lower degree. A suitable component of Laplace-Lenz vector is used in place of the energy.]

[10] S. Herrick (1971): '*Astrodynamics*', Vol.1, Van Nostrand Reinhold, London [The derivation of Gibbs' formulae for the computation of the velocities can be found here.]

[11] P. S. Laplace (1780): Mém. Acad. R. Sci. Paris, in *Laplace's collected works* , 93–146 [Source reference for Laplace's method.]

[12] A. M. Legendre (1805): '*Nouvelles M´ethodes pour la d´eterminations des orbits des comètes*', ed. F. Didot [This treatise is related to the dispute for the invention of the least squares method.]'

[13] A. Milani (1999): 'The Asteroid Identification Problem I: recovery of lost asteroids', *Icarus* , 269–292 [Description of the orbit identification problems.]

[14] A. Milani, G. F. Gronchi, M. de' Michieli Vitturi and Z. Knežević (2004): 'Orbit Determination with Very Short Arcs. I Admissible Regions', *CMDA* , 59–87 [Here the admissible region is introduced, together with the detailed proofs of its properties.]

[15] A. Milani, A., G. F. Gronchi, Z. Knežević, M. E. Sansaturio and O. Arratia (2005): 'Orbit Determination with Very Short Arcs. II Identifications', *Icarus* , 350–374 [Introduction of the recursive attribution method, that employs the admissible region.]

[16] A. Milani, A. La Spina, M. E. Sansaturio and S. R. Chesley (2000): 'The Asteroid Identification Problem III. Proposing identifications', *Icarus* , 39–53 [Introduction of an orbit identification method.]

[17] A. Milani, S. R. Chesley, M. E. Sansaturio, G. Tommei, and G. B. Valsecchi (2005): 'Nonlinear impact monitoring: line of variation searches for impactors', *Icarus* , 362–384 [Introduction of the Line of Variation.]

[18] A. Milani, G. F. Gronchi, D. Farnocchia, Z. Knežević, R. Jedicke, L. Dennau, F. Pierfederici (2008): 'Topocentric Orbit Determination: Algorithms for the Next Generation Surveys', *Icarus* , 474–492 [Comparison of the classical methods, showing an extension of Laplace's method, that accounts for topocentric observations. There are also examples of occurrence of alternative solutions.]

[19] A. Milani, G. F. Gronchi (2010): '*Theory of Orbit Determination*', Cambridge University Press [Treatise about the modern methods of orbit determination. It describes in details most of the issues that we have presented.]

[20] O. F. Mossotti (1816-1818): 'Nuova analisi del problema di determinare le orbite dei corpi celesti', in *Mossotti, scritti*, Domus Galileiana, Pisa [Articles with some classical orbit determination methods, different from the ones by Gauss and Laplace.]

[21] H. C. Plummer (1960): '*An introductory treatise on Dynamical Astronomy*', Cambridge University press, reprinted by Dover publications, New York [In this book there is a detailed review of Charlier's theory.]

[22] H. Poincaré (1906): 'Sur la détermination des orbites par la méthode de Laplace', *Bulletin astronomique* , 161–187 [It is about the use of Laplace's method, and introduces also a way to deal with topocentric observations.]

[23] S. Stigler (1981): 'Gauss and the Invention of Least Squares', *Ann. of Stat.* /3 , 465–474 [The history of the dispute for the invention of the least squares method.]

[24] L. G. Taff, D. L. Hall (1977): 'The use of angles and angular rates. I - Initial orbit determination', *CMDA* , 481–488 [The equations for the linkage using the angular momentum and energy integrals are first introduced here.]

[25] F. Tisserand, J. Perchot (1899): '*Lecons sur la détermination des orbites*', Paris, Gauthier-Villars [Important treatise of orbit determination.]

Biographical Sketch

Giovanni Federico Gronchi received his Ph.D. in Mathematics from the University of Pisa. At present he is Full Professor of Mathematical Physics at the Department of Mathematics, University of Pisa. His research is about Solar System body dynamics, perturbation theory, orbit determination, singularities and periodic orbits of the N-body problem.